# THE ARITHMETIZING OF MATHEMATICS.*

*An Address delivered at the public meeting of the Royal Academy of Sciences of Göttingen, November 2nd, 1895.*

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THOUGH the details of mathematical science, by their very nature, elude the comprehension of the layman, and therefore fail to arouse his interest, yet the mathematician may profitably indicate certain general points of view from which he surveys his science, especially if these points of view determine his attitude to kindred subjects. I propose therefore on the present occasion to explain my position in regard to an important mathematical tendency which has as its chief exponent Weierstrass, whose eightieth birthday we have lately celebrated. I refer to the *arithmetizing* of mathematics. Some account of this tendency and its origin may be given by way of preface.

The popular conception of mathematics is that of a strictly logically coördinated system, complete in itself, such as we meet with, for instance, in Euclid's geometry; but as a matter of fact, modern mathematics in its origin was of a totally different character. With the contemplation of nature as starting point, and its interpretation as object, a philosophical principle, the principle of continuity, was made fundamental; and the use of this principle characterizes the work of the great pioneers, Newton and Leibnitz, and the mathematicians of the whole of the eighteenth century—a century of discoveries in the evolution of mathematics. Gradually, however, a more critical spirit asserted itself and demanded a logical justification for the innovations made with such assurance, the establishment, as it were, of law and order after the long and victorious campaign. This was the time of Gauss and Abel, of Cauchy and Dirichlet. But this was not the end of the matter. Gauss, taking for granted the continuity of space, unhesitatingly used space intuition as a basis for his proofs; but closer investigation showed not only that many special points still needed proof, but also that space intuition had led to the too hasty assumption of the generality of certain theorems which are by no means general. Hence arose the demand for exclusively arithmetical methods of proof; nothing shall be

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accepted as a part of the science unless its rigorous truth can be clearly demonstrated by the ordinary operations of analysis. A glance at the more modern text-books of the differential and integral calculus suffices to show the great change in method; where formerly a diagram served as proof, we now find continual discussions of quantities which become smaller than, or which can be taken smaller than, any given small quantity. The continuity of a variable, and what it implies, or does not imply, are discussed, and a question is brought forward whether we can, properly speaking, differentiate or integrate a function at all. This is the Weierstrassian method in mathematics, the "Weierstrass'sche Strenge," as it is called.

Of course even this assigns no absolute standard of exactness; we can introduce further refinements if still stricter limitations are placed on the association of the quantities. This is exemplified in Kronecker's refusal to employ irrational numbers, and consequent reduction of mathematics to relations between whole numbers only; and in another way in the efforts made to introduce symbols for the different logical processes, in order to get rid of the association of ideas, and the lack of accuracy which creeps in unnoticed, and therefore not allowed for, when ordinary language is used. In this connection special mention must be made of an Italian mathematician, Peano, of Turin, to whom we are indebted for various interesting notes on other points.

Summing up all these developments in the phrase, *the arithmetizing of mathematics*, I pass on to consider the influence of the tendency here described on parts of the science outside the range of analysis proper. Thus, as you see, while voluntarily acknowledging the exceptional importance of the tendency, I do not grant that the arithmetized science is the essence of mathematics; and my remarks have therefore the two-fold character of positive approbation, and negative disapproval. For since I consider that the essential point is not the mere putting of the argument into the arithmetical form, but the more rigid logic obtained by means of this form, it seems to me desirable—and this is the positive side of my thesis—to subject the remaining divisions of mathematics to a fresh investigation based on the arithmetical foundation of analysis. On the other hand I have to point out most emphatically—and this is the negative part of my task—that it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, intuition has its special province. For the sake of completeness I ought also to deal with the algorith-

mic side of mathematics, discussing the importance of symbolic methods, but as this subject does not appeal to me personally, I shall not enter upon it. It must be understood that I have not much that is new to say on special points; my object is rather to collect and arrange material already familiar, justifying its existence where necessary.

In the short time at my disposal I must content myself with presenting the most important points; I begin therefore by tracing the relation of the positive part of my thesis to the domain of geometry. The arithmetizing of mathematics began originally, as I pointed out, by ousting space intuition; the first problem that confronts us as we turn to geometry is therefore that of reconciling the results obtained by arithmetical methods with our conception of space. By this I mean that we accept the ordinary principles of analytical geometry, and try to find from these the geometrical interpretation of the more modern analytical developments. This problem, while presenting no special difficulty, has yet many ramifications, as I have had the opportunity of showing during the past year in a seminar devoted to this subject. The net result is, on the one hand, a refinement of the process of space intuition; and on the other, an advantage due to the clearer view that is hereby obtained of the analytical results considered, with the consequent elimination of the paradoxical character that is otherwise apt to attach itself to them. What is the most general idea of a curve, of a surface? What is meant by saying of a curve, etc., that it is "analytic" or "non-analytic?" These and similar questions must be thoroughly sifted and clearly explained. The next point is that we must subject the fundamental principles of geometry to a fresh investigation. As far as the theory of the matter is concerned, this might very well be done, as it was originally, on purely geometrical lines; but in practice on account of the overwhelming complications that present themselves, recourse must be had to the processes of analysis, that is to the methods of analytical geometry. The investigation of the formulæ by means of which we represent the different forms in space (that is, the so-called non-Euclidian geometry, and all that is connected with it) disposes of only one side, and that the more obvious one, of the inquiry; there still remains the more important question: What justification have we for regarding the totality of points in space as a number-manifoldness in which we interpolate the irrational numbers in the usual manner between the rational numbers arranged in three directions? We ultimately perceive that space intui-

tion is an inexact conception, and that in order that we may subject it to mathematical treatment, we idealize it by means of the so-called axioms, which actually serve as postulates. Kerry, who died at an early age, dealt with the philosophical side of these questions, and I agree with his results in the main and especially as regards his criticism of DuBois Reymond. Conversely this fresh determination of our conception of space has in its turn given rise to new refinements of our analytical ideas. We picture before us in space an infinite number of points and forms composed of them ; from this idea have sprung the fundamental investigations on masses of points and transfinite numbers with which G. Cantor has opened up new spheres of thought to arithmetical science. Finally it is much to be desired that full use should be made of the new point of view in the further exposition of geometry, especially infinitesimal geometry ; this result will be most easily attained by treating the subject analytically. Of course, I do not mean by this a blind calculation with $x$, $y$ and $z$, but merely a subsidiary use of these quantities when the question concerns the precise determination of boundary conditions.

From this outline of the new geometrical programme you see that it differs greatly from any that was accepted during the first half of this century, when the prevailing tendencies led to the development of projective geometry, which has long been established as a permanent constituent of our subject. Projective geometry has opened up for us with the greatest ease many new tracts of country in our science, and has been rightly called a royal road to its own particular branch of learning; our new road is on the contrary arduous and thorny, and unremitting care is needed to clear a way through the obstacles. It leads us back to what is more nearly the geometry of the ancients, and in the light of our modern ideas we learn to understand precisely the true nature of the latter, as Zeuthen has lately shown in the most brilliant manner.

Moreover we must introduce the same process of reasoning into mechanics and mathematical physics. To avoid going too much into detail I will merely explain this by two examples. Throughout applied mathematics, as in the case of space intuition, we must idealize natural objects before we can use them for purposes of mathematical argument; but we find continually that in one and the same subject we may idealize objects in different ways, according to the purpose that we have in view. To mention only a single instance,

we treat matter either as continuous throughout space, or as made up of separate molecules, which we may consider to be either at rest or in rapid motion. How and to what degree are these different hypotheses equivalent in regard to the mathematical consequences that can be deduced from them? The earlier expositions of Poisson and others, as also the developments of the Kinetic Theory of Gases, are not sufficiently thorough in this respect for the modern mathematician; the problem requires to be investigated afresh ab initio. I expect that a publication by Boltzmann, which will shortly appear, will contain some interesting conclusions on this subject.

Another question is this: Practical physics provides us plentifully with experimental results, which we unconsciously generalize and adopt as theorems about the idealized objects. The existence of the so-called Green's function on any closed surface with an arbitrarily chosen pole, corresponding to the fact in electricity that a conductor under the influence of a charged point is in a state of electrical equilibrium, belongs to this category; as also the theorem that every finite elastic body is capable of an infinite series of harmonic oscillations, and my deduction of the fundamental propositions of Riemann's Theory of Abelian Functions from our knowledge of the electric currents started on any conductor when the poles of a galvanic battery are applied to it. Are these indeed, taken in the abstract, exact mathematical theorems, or how must they be limited and defined in order that they may become so? Mathematicians have successfully sought to investigate this; first, C. Neumann and Schwarz with their theory of Potential, and later the French school, following on the lines of the German, with the result that the theorems taken from physics have been shown to hold good to a very considerable extent. You see here what is the precise object of these renewed investigations; not any new physical insight, but abstract mathematical argument in itself, on account of the clearness and precision which will thereby be added to our view of the experimental facts. If I may use an expression of Jacobi's in a somewhat modified sense, it is merely a question of intellectual integrity, "die Ehre des menschlichen Geistes."

After expressing myself thus it is not easy, without running counter to the foregoing conclusions, to secure to intuition her due share in our science; and yet it is exactly on this antithesis that the point of my present statements depends. I am now thinking not so much of the cultivated intuition just discussed, which has been developed under

the influence of logical deduction and might almost be called a form of memory; but rather of the naive intuition, largely a natural gift, which is unconsciously increased by minute study of one branch or other of the science. The word intuition (Anschauung) is perhaps not well chosen; I mean it to include that instinctive feeling for the proportion of the moving parts with which the engineer criticises the distribution of power in any piece of mechanism he has constructed; and even that indefinite conviction the practiced calculator possesses as to the convergence of any infinite process that lies before him. I maintain that mathematical intuition—so understood—is always far in advance of logical reasoning and covers a wider field.

I might now introduce a historical excursus, showing that in the development of most of the branches of our science, intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole, but also of many subjects that have come into existence only in the present century. For example, I may remind you of Riemann's Theory of the Functions of a Complex Variable; and I am glad to add also that the Theory of Numbers, a subject which for a long time seemed to be most unsuited for intuitive methods of treatment, appears to have received a fresh impetus from the application of intuition in the hands of Minkowski and others. After this it would be a matter of great interest to trace from the present standpoint the development, not of particular mathematical subjects, but of the individual mathematician; but in regard to this it must suffice to mention that the two most active mathematical investigators of the present day, Lie in Leipzig, and Poincaré in Paris, both originally made use of intuitive methods. But all this, if I pursued it further, would lead us too much into detail, and finally bring us only to particular cases. I prefer to sketch the every day results of this somewhat refined intuition, as regards the quantitative, rather than the merely arithmetical or constructive, treatment of physical or technical problems. Let me refer again to the two examples from the theory of electricity already adduced; any physicist would be able to trace, without further difficulty, and with tolerable accuracy, the form of the surface of Green's function, or, in the second experiment, the shape of the lines of force in a given case. Again, consider any given differential equation, I will say, to take the most simple instance, a differential equation of the first order in two variables. Most probably the analyti-

cal method of solution fails; nevertheless we can at once find graphically the general form of the integral curves, as has recently been done for the renowned differential equation of the Problem of Three Bodies by Lord Kelvin, a master in the art of mathematical intuition. The question in all such cases, to use the language of analysis, is one of interpolation, in which less stress is laid on exactness in particular details than on a consideration of the general conditions. I will once more emphasize the fact that in stating all our laws of nature, or in trying to formulate mathematically any actual occurrence, the art lies in making a similar use of interpolation; for we have to consider the simple laws connecting the essential quantities, apart from the multitude of fortuitous disturbances. This is ultimately what I have termed above the process of idealization. Logical investigation is not in place until intuition has completed the task of idealization.

I beg that you will consider these remarks as a description, not as an explanation, of what actually occurs. The mathematician can do no more than state the character of each particular psychical operation from observations of his own mental process. Perhaps some day physiology and experimental psychology will enable us to draw more accurate conclusions as to the relation between the processes of intuition and those of logical thought. The great differences shown by observations of different individuals confirm the supposition that it is indeed a question of distinct, that is, not necessarily connected, mental activities. Modern psychologists distinguish between visual, motor and auditory endowments; mathematical intuition, as above defined, appears to belong more closely to the first two classes, and the logical method to the third class. In common with many of my fellow mathematicians I gladly welcome these investigations which psychologists have only just undertaken, for it is to be hoped that with the increase of accurate information about the psychological conditions of mathematical thought and their particular varieties, many of the differences of opinion which necessarily remain unsettled at present will disappear.

I must add a few words on mathematics from the point of view of pedagogy. We observe in Germany at the present day a very remarkable condition of affairs in this respect; two opposing currents run side by side without affecting one another appreciably. Among the teachers in our Gymnasia the need of mathematical instruction based on intuitive methods has now been so strongly and universally empha-

sized that one is compelled to enter a protest, and vigorously insist on the necessity for strict logical treatment. This is the central thought of a small pamphlet on elementary geometrical problems which I published last summer. Among the university professors of our subject exactly the reverse is the case; intuition is frequently not only undervalued, but as much as possible ignored. This is doubtless a consequence of the intrinsic importance of the arithmetizing tendency in modern mathematics. But the result reaches far beyond the mark. It is high time to assert openly once for all that this implies, not only a false pedagogy, but also a distorted view of the science. I gladly yield the utmost freedom to the preferences of individual academic teachers, and have always discouraged the laying-down of general rules for higher mathematical teaching, but this shall not prevent me from saying that two classes at least of mathematical lectures must be based on intuition; the elementary lectures which actually introduce the beginner to higher mathematics—for the scholar must naturally follow the same course of development on a smaller scale, that the science itself has taken on a larger—and the lectures which are intended for those whose work is largely done by intuitive methods, namely, natural scientists and engineers. Through this one-sided adherence to logical form we have lost among these classes of men much of the prestige properly belonging to mathematics, and it is a pressing and urgent duty to regain this prestige by judicious treatment.

To return to theoretical considerations, the general views which I uphold in regard to the present problems of mathematical science need scarcely be specially formulated. While I desire in every case the fullest logical working out of the material, yet I demand at the same time an intuitive grasp and investigation of the subject from all sides. Mathematical developments originating in intuition must not be considered actual constituents of the science till they have been brought into a strictly logical form. Conversely, the mere abstract statement of logical relations cannot satisfy us until the extent of their application to every branch of intuition is vividly set forth, and we recognize the manifold connections of the logical scheme, depending on the branch which we have chosen, to the other divisions of our knowledge. The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the

branches as its essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts.

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## DARBOUX'S MEMOIR ON CYCLIQUES.

*Sur une Classe Remarquable de Courbes et de Surfaces Algébriques et sur la théorie des Imaginaires*, par GASTON DARBOUX, Doyen de la Faculté des Sciences de Paris. Second tirage. Paris, A. HERMANN, 1896.

IT is a pity that M. Darboux did not make some additions to this work since its first publication in 1873. He is so full of ideas in a number of mathematical directions that it is a cause for regret that he has not devoted some more time to a subject which offered him once such a fruitful field for original investigation. However, those who know the work will be glad to make acquaintance with it again, and others who are tired of conic sections and quadrics may be gratified by finding in it a somewhat similar but still novel field of investigation.

Darboux calls the curves and surfaces which he treats of cycliques and cyclides, respectively. The latter word has been generally adopted in English as a name for the surfaces, but the former has been replaced by bicircular quartics and sphero-quartics, which two designations conveniently distinguish between the plane and spherical curves. The name cyclide had already been used for a particular form of the surface by Dupin, and soon came to be adopted generally in its present sense, while the different names of the curves arose from their being studied independently in Great Britain and France. At one time, from about 1865 until early in the seventies, these curves and surfaces were studied enthusiastically in France by Darboux, Laguerre, de la Gournerie, Moutard, Mannheim and others, while they received attention in England at the hands of Crofton and Clifford, and in Ireland secured very full and adequate treatment from Casey. All these mathematicians worked at the same time and nearly entirely independently of each other. Thus Darboux's book contains much that has been more fully worked out elsewhere, and, besides, of course, it has no references to many striking results that have been arrived at in this branch of geometry, as, for instance, Casey's ingenious method of rectification of the bicircular quartics and sphero-quartics. But these wants are made up

for by Darboux's methods of investigation and by his curious paths of inquiry, from which new views and sidelights on the subject are continually obtained. It is this, as well as the incidental solutions of many problems not directly connected with the subject, that gives a charm to the work, and only those who have a preference for formal and logically ordered treatises will be displeased with its discursive character.

M. Darboux begins by considering the transformation of inversion, and in addition to the ordinary properties of this process shows that foci and focal curves, when inverted, possess the same relation to the inverted curve. Then he shows that every *cyclique* is transformed into a similar curve by inversion. Thus for an origin outside of its plane, the bicircular quartic, or plane quartic with the two circular points at infinity as nodes, is transformed into a sphero-quartic, or curve of intersection of a sphere and a quadric. In this way he is enabled to select the most suitable curves for his first investigation. Choosing the sphero-quartic he makes use of the known result that four cones of the second order can be described passing through the curve. Since the tangent planes of the cones intersect the sphere in circles having double contact with the curve, he thus deduces the four-fold generation of the sphero-quartic as the envelope of small circles orthogonal to a given circle J and having their centres on a sphero-conic F. Then by geometrical considerations the four circles such as J are shown to be mutually orthogonal and the four sphero-conics such as H to be confocal, and further the focal curves are shown to lie on four mutually orthogonal spheres having their centres at the vertices of the cones. In this way M. Darboux establishes these and many other properties of the curves by means of previously known results. M. Darboux obtains a number of properties of the bicircular quartics, sphero-quartics and allied curves of higher orders by the use of imaginaries. In particular we may mention the differential equation of the system of confocal bicircular quartics, viz:

$$\frac{du}{\sqrt{f(u)}} \pm \frac{dv}{\sqrt{f_1(v)}} = 0.$$

where

$$f(u) = A(u-a_1)(u-a_2)(u-a_3)(u-a_4)$$
$$f_1(v) = B(v-b_1)(v-b_2)(v-b_3)(v-b_4),$$
$$u = x+iy, \quad v = x-iy,$$

and the anharmonic ratios of

$$f(u)=0,\quad f_1(v)=0$$

are equal.

This equation gives the complete theory of the arrangement of the 16 foci of the curve, for by the definition of a focus these points are found from $du=0$, $dv=0$. Thus we get $f(u)=0, f_1(v)=0$, which determine 16 points; and these from the equality of the anharmonic ratios must lie by fours on four circles.

By the use of imaginaries M. Darboux also obtains properties of curves comprised in the equation

$$R_1 R_2 R_3 \cdots = k\, r_1 r_2 r_3 \cdots ,$$

where $R_1, R_2, \cdots, r_1, r_2 \cdots$, are the distances from fixed points, and he shows that such curves can be written in this form in an infinite number of ways. Similar results are obtained for the sphere, and theorems concerning closed polygons formed by the imaginary generators of the sphere are proved. He thus gives a new demonstration of Poncelet's theorems for polygons of an even number of sides and arrives at an important theorem concerning polyhedra formed by generators of a quadric, viz: that if they form a polygon having its vertices on the curve of intersection with another quadric, then there exist an infinite number of polygons formed in the same way. M. Darboux observes that this result was given by Moutard in the *Nouvelles Annales de Mathématiques* in 1864. I think, however, that Steiner and Hart were acquainted with it previously. In 1846 Steiner gave in Crelle's Journal a theorem concerning polygons inscribed in a binodal quartic. This is to the effect that, if a certain condition is satisfied, an infinite number of polygons can be described, such that one set of alternate sides all pass through one node, while the other set of alternate sides pass through the other node. Now let the nodes be the circular points, then if we invert from a point outside the plane, the lines through the nodes become the imaginary generators of a sphere, and the curve becomes a sphero-quartic. Thus we see the identity of Steiner's theorem with that just mentioned. I may notice that Professor A. R. Forsyth, who has treated this problem from an analytical point of view, makes no mention of those who had gone before him. (*Proceedings of the London Mathematical Society*, vol. 14, p. 35.)

M. Darboux further notices that the same property exists when we substitute for the circumscribing quadric quadrics

inscribed in the developable formed by the inscribed quadric and one of the circumscribing quadrics, or which is the same thing, if we suppose the vertices to lie on confocal quadrics; and he observes that this proposition seems new. Hart, however, was previously aware of the fact that there could be an infinite number of geodesic polygons circumscribed about a line of curvature of a quadric so that these vertices lie on other lines of curvature. Now, if we suppose the quadric passing through the inscribed line of curvature to coincide with the given one, the geodesics become simply the generators. Thus Darboux's result is comprised in Hart's theorem. (*Cambridge and Dublin Mathematical Journal*, vol. 4, p. 192.)

Proceeding to consider the surface M. Darboux makes use of analysis. With the help of the inscribed quadrics, viz.: quadrics such as $V$ when the equation of the surface takes the form $S^2=V$, where $S$ is a sphere, he deduces the generation of the cyclide as the envelope of a sphere orthogonal to a given sphere $J$ and having its centre on a quadric $F$. Further he shows that this generation is fivefold, the five spheres such as $J$ being mutually orthogonal, and the five quadrics such as $F$ being confocal, while the five sphero-quartics $FJ$ are the focal curves. This mode of treatment, it may be observed, has been adopted by Dr. Salmon in his geometry of three dimensions. The remarkable form assumed by the equation of confocal cyclides in terms of the five spheres is worth noticing. If $X$, $Y$, $Z$, $U$, $V$ represent the squares of the tangents drawn from a point to the spheres divided by the radius of each sphere, then $X^2+Y^2+Z^2+U^2+V^2=0$ identically; and the equations

$$\frac{X^2}{\lambda-a_1}+\frac{Y^2}{\lambda-a_2}+\frac{Z^2}{\lambda-a_3}+\frac{U^2}{\lambda-a_4}+\frac{V^2}{\lambda-a_5}=0$$

represent confocal cyclides. The remarkable fact that spheres such as $l_i X+m_i Y+n_i Z+p_i U+q_i V=0$, cut orthogonally if $\Sigma l_1 l_2=0$, shows at once that the three confocal cyclides drawn through a point cut orthogonally. There is in these equations an analogy to linear expressions passing through a point and cones, if we may call them so, in four dimensions. In fact just as bicircular quartics are the inverses of sphero-conics, so cyclides may be regarded as the inverses of the sphero-quadrics of four dimensions. This system of coördinates enables us to express the coördinates of a point on a cyclide in terms of two parameters. For solving for $X$ in terms of the parameters of three surfaces of the system we get

$$X^2=\frac{(a_1-\lambda_1)(a-\lambda_2)(a_1-\lambda_3)}{f'(a_1)},$$

where

$$f(\lambda)=(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)(\lambda-a_5),$$

and similar values for $Y$, $Z$, $U$, $V$. Hence if we suppose $\lambda_3$ to be a constant for a given surface we obtain, from

$$X=\frac{1}{r}(x^2+y^2+z^2-2\alpha x+2\beta y-2\gamma^2+\delta)$$

and similar equations, $x, y, z$ expressed linearly in terms of the five radicals $\sqrt{\{(a_i-\lambda_1)(a_i-\lambda_2)\}}$. Thus by the theory of hyperelliptic integrals if we write

$$\frac{d\lambda_1}{\sqrt{f(\lambda_1)}}+\frac{d\lambda_2}{\sqrt{f(\lambda_2)}}=du_1,\quad \frac{\lambda_1 d\lambda_1}{\sqrt{f(\lambda_1)}}+\frac{\lambda_2 d\lambda_2}{\sqrt{f(\lambda_2)}}=du_2,$$

we have the coördinates expressed linearly in terms of the inverse hyperelliptic functions of $u, v$, called $al(u_1, u_2)_1$, $al(u_1, u_2)_2$, etc., by Weierstrass, just as the coördinates of a point on a quadriquadric curve can be expressed linearly in terms of $\mathrm{sn}u$, $\mathrm{cn}u$, $\mathrm{dn}u$.

In the notes and additions, which comprise a considerable portion of the work, M. Darboux gives a number of miscellaneous theorems concerning curves and surfaces. In particular he studies the system of coördinates obtained by taking as variables the parameters of the tangents which can be drawn from a point to a given conic. Thus if the conic is $\beta^2-4\alpha\gamma=0$, a tangent may be written

$$\alpha\rho^2+\beta\rho+\gamma=0,$$

so that if $\rho_1, \rho_2$ are the parameters of the tangents drawn from $\alpha, \beta, \gamma$, we have

$$\alpha=-\frac{\beta}{\rho_1+\rho_2}=\frac{\gamma}{\rho_1\rho_2}.$$

By means of these coördinates he obtains several interesting results regarding polygons. For instance, if a curve of the $n^{\text{th}}$ degree pass through the intersection of the $n$ tangents, $A_1, A_2 \cdots A_n$ with the $n$ tangents $B_1, B_2, \cdots B_n$, its equation is

$$A_1 A_2 \cdots A_n = k B_1 B_2 \cdots B_n.$$

Since

$$A_i=\alpha a_i^2+\beta a_i+\gamma=\alpha(a_i-\rho_1)(a_i-\rho_2) \text{ and } B_i=\alpha(b_i-\rho_1)(b_i-\rho_2),$$

this equation takes the form

$$\frac{\varphi(\rho_1)\varphi(\rho_2)}{\psi(\rho_1)\psi(\rho_2)}=k.$$

But this form can be assumed in a singly infinite number of other ways, as we see by writing it

$$\frac{\{\psi(\rho_1)+\lambda\varphi(\rho_1)\}\ \{\psi(\rho_2)+\lambda\psi(\rho_2)\}}{\{k\lambda\psi(\rho_1)+\varphi(\rho_1)\}\ \{k\lambda\psi(\rho_2)+\varphi(\rho_2)\}}=\frac{1}{k}$$

Hence if a curve pass through the $n^2$ points where $n$ tangents to a conic meet other $n$ tangents, it must pass through an infinity of such systems of $n^2$ points. Further, if a curve of the $n^{\text{th}}$ degree pass through all the points of intersection of $n+1$ tangents to the conic, it may be written

$$\frac{a_1}{A_1}+\frac{a_2}{A_2}+\cdots\frac{a_{n+1}}{A_{n+1}}=0$$

which becomes

$$\Sigma\frac{a_i}{(a_i-\rho_1)\ (a_i-\rho_2)}=0$$

or, if we multiply by $\rho_1-\rho_2$

$$\Sigma\frac{a_i}{a_i-\rho_1}=\Sigma\frac{a_i}{a_i-\rho_2}$$

that is

$$\frac{f(\rho_1)}{\varphi(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)}$$

which may also be written

$$\frac{f(\rho_1)}{\varphi(\rho_1)+kf(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)+kf(\rho_2)},$$

which is of the same form as the preceding with indeterminate $k$. Hence the curve will pass through all the vertices of an infinite number of polygons of $n+1$ sides circumscribed to the conic.

In this way M. Darboux proves the theorem originally obtained by Lüroth (*Math. Annalen* vol. 1.) viz: that if a quartic curve pass through all the points of intersection of five lines it cannot be the general curve of that degree, but must satisfy an invariant relation. M. Darboux might have noticed also that a general quartic curve cannot pass through the 16 points where four tangents of a conic meet four other tangents.* The invariant condition satisfied in this case is a problem suggested here for solution. Again by the use of these coördinates M. Darboux gives a very simple proof of Poncelet's theorem concerning polygons.

* It may be observed that this result can also be put in the form: the twelve vertices and eight points of contact of two quadrilaterals circumscribed about a conic lie on a quartic curve which is not a general one.

It may be noticed that W. K. Clifford arrived at these results in precisely the same way (*Proceedings of the London Mathematical Society*, vol. 7, p. 29), and when he found that he had been anticipated by Darboux, expressed his opinion that this, viz: the work of Darboux that we are now considering, is a book which it is almost inexcusable in a geometer not to have read, marked, learned and inwardly digested.

There are many other interesting investigations in the book, especially a study at considerable length of the intersection of a sphere with a cyclide; an extension also of Ivory's theorem to confocal cyclides is worthy of notice. If $A$, $B$, $C$ are three points on the cyclide and $A'$, $B'$, $C'$ are three corresponding points on a confocal cyclide, M. Darboux shows that the relation

$$AB'.BC'.CB'=BA'.CB'.BC'$$

connects the distances between the points.

R. A. ROBERTS.

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## BESSEL FUNCTIONS.

*A Treatise on Bessel Functions and their Applications to Physics.* By ANDREW GRAY and G. B. MATHEWS. Macmillan & Co. 1895. 8vo, x+292 pp.

The transcendental functions to which Bessel's name has been attached are not only of the highest importance in mathematical physics, second perhaps only to the trigonometric and exponential functions, they are also of great interest to the student of pure mathematics both from the formal side and from the point of view of the theory of functions. There has, however, up to this time been no connected treatment of these functions in the English language, with the exception of the utterly inadequate treatment contained in the last sixty-five pages of Todhunter's book, *The Functions of Laplace, Lamé and Bessel*, published twenty years ago. The German monographs by C. Neumann and Lommel make no attempt to cover more than small portions of the subject, and the same is true to an even greater extent of the sections devoted to Bessel's functions in Heine's *Kugelfunctionen*, Basset's *Hydrodynamics*, Rayleigh's *Sound* and elsewhere. Messrs. Gray and Mathews have therefore filled a real gap in mathematical literature.

The authors make it clear in their preface that their own

interest is chiefly for the applications to mathematical physics. They have, however, none of the intolerance for and dislike of pure mathematics which is unfortunately too often manifested. The authors' position may, perhaps, be best explained in their own words: "Some readers may be inclined to think that the earlier chapters contain a needless amount of tedious analysis. . . . . As a matter of fact it will be found that little, if any, of the analytical theory included in the present work has failed to be of some use or other in the later chapters; and we are so far from thinking that anything superfluous has been inserted that we could almost wish that space would have allowed of a more extended treatment, especially in the chapters on the complex theory and on definite integrals."

The book consists of fifteen chapters, followed by a few pages of miscellaneous matter and tables. Of the fifteen chapters, eight (87 pages) are devoted to the theory, while the remaining seven (135 pages) treat of the applications. This separation of theory and application is by no means complete, much analysis which might easily have been included in the earlier chapters being left until needed, while a few physical problems are introduced for the sake of illustration into the chapters on the theory of Bessel's functions. We think there can be no doubt that the authors have been wise in carrying this separation at least as far as they have done, as it conduces greatly to the clearness of the book, and makes it, what is very important, a convenient book of reference. The book might have been made even more useful in this respect than it is now by a somewhat extended collection of formulæ, including in particular a table of definite integrals involving Bessel's functions, which could have been placed with the numerical tables at the end of the book.

CHAPTER I: *Introductory* shows how Bessel's functions naturally present themselves in two simple physical and in one astronomical problem. The problems chosen (vibration of a heavy chain hanging from one end, cooling of a cylinder, Kepler's equation) are not only interesting in themselves; they have also a decided historic interest as being among the first problems in which Bessel's functions were used.*

* There is a slight historical slip on p. 2, in the statement that after the problem of the heavy chain "the next appearance of a Bessel function is in Fourier's *Théorie analytique de la chaleur.*" The problem of the vibration of a circular membrane had been treated by Euler in 1764 (Cf. *Bulletin*, February, 1893, p. 108). A reference to Bessel's paper of 1818 might also well have been made.

CHAPTER II: *Solution of the Differential Equation.* The title of this chapter indicates the starting point chosen by the authors for the systematic study of Bessel's functions. The differential equation is solved by means of power series and a few simple relations between the Bessel's functions thus obtained are established. On p. 8 it is pointed out that the series for $J_{-n}(x)$ ceases to exist ("becomes unintelligible" is the unfortunate expression used) when $n$ is a positive integer, and on p. 12 the function is defined in this case by the formula: $J_{-n}(x)=(-1)^n J_n(x)$, even the mere statement of the reason for this definition (viz.: to preserve the continuity with regard to $n$) being relegated to a footnote. The treatment of such a fundamental point as this, which would not have covered half a page, ought surely to have been given. We are glad to find the Bessel's function of the second kind* $Y_n(x)$ introduced here by means of power series; the interesting, but for the beginner artificial, form in terms of Bessel's functions of the first kind is given later (pp. 21–23).

CHAPTER III: *Functions of Integral Order. Expansions in series of Bessel Functions.*—This chapter opens with Schlömilch's elegant definition of Bessel's functions of the first kind of integral orders as the coefficients in the expansion of $e^{\frac{x}{2}(t-1/t)}$ according to ascending and descending powers of $t$. Here as elsewhere in the book the authors have adopted a half-way course with regard to mathematical rigor which can perhaps be justified when we consider the class of readers for which the book is written. Thus they usually establish the convergency of the series or integrals with which they deal, but leave the question as to whether the series or integrals converge to the right value, and also questions as to rearranging the terms of a series, differentiating a series, etc., almost, if not quite, unconsidered. It should, however, be added that the authors sometimes regard this method of obtaining results as merely heuristic and establish their results when once found by other methods.

CHAPTER IV: *Semiconvergent Expansions* might well have been postponed until later, as at this point only Bessel's functions of the first kind can be treated at all, and that in a very imperfect way. The term *semiconvergent* is here used in its proper sense, a series of this sort being divergent, but such that for certain values of the argument and of $n$

* The authors have seen fit to attach Neumann's name to this function, although in the case $n=0$ the function had been used by Euler.

the sum of its first $n$ terms gives an approximation to the function we wish to represent. The use of the word in this sense is more than forty years old, and it is greatly to be regretted that of late years the term has been used in English in a totally different sense, viz.: non-absolutely or conditionally convergent. After the general solution $u_r$ of Bessel's equation has been developed into a semiconvergent series the statement is made (p. 37): "It is to be observed ..... that it is not to be inferred that the expression above given for $u_r$ reflects in any adequate way the *functional* properties of an exact solution ..... These considerations, however, do not debar us from employing the expressions $u_r$ up to a certain point for approximate numerical calculation ....." The distinction here made between the numerical values of a function and its "functional properties" is one which would, we think, be found in the minds of a very large number of people. There exists, however, no such distinction in reality, as a function is nothing more nor less than what might be called an ideal table of numerical values, and any properties of a function are merely propositions concerning these numerical values. The peculiarity of a semiconvergent series is that it does not permit us, like the ordinary representations of functions, to compute these numerical values to any required degree of accuracy, but merely to a certain limited degree of accuracy. The subject of semiconvergent series together with the allied subject of the asymptotic values of functions is one with which most readers of this *Treatise* are sure to be so unfamiliar, and it is at the same time a subject of such importance both practically and theoretically, that we think a more extended and more accurate treatment might well have been given of it.

CHAPTER V: *The Zeros of the Bessel Functions* gives far too brief a treatment (a little over six pages) of one of the most important and fascinating questions connected with Bessel's functions. The chapter opens with Bessel's original proof that $J_0(x)$ has an infinite number of real roots, this proof being reproduced "on account not only of its historical interest, but of its directness and simplicity." The fact that the considerations here given not only establish the existence of an infinite number of real roots, but also serve to completely separate them, is not, however, touched upon. Stokes's method of computing the large roots of $J_0(x)$ by means of semiconvergent series is also treated in this chapter, but no discussion is given of the degree of accuracy which it enables us to attain. Another

matter which might well have found a place, is the application of the method developed by Sturm,* or of some suitable modification of it† to Bessel's equation. At least a reference should have been made to Hurwitz's paper in Math. Ann., vol. 33.

CHAPTER VI: *Fourier-Bessel Expansions.* Certain definite integral formulæ are first deduced by means of Green's theorem. Then follows a simple problem in the flow of heat which leads up to the development of an arbitrary function in the form: $\sum_{s=1}^{s=\infty} A_s J_0(\lambda_s r)$ where the quantities $\lambda_s$ are the roots of the equation $\lambda k J_0'(\lambda)+h J_0(\lambda)=0$. Other developments of a similar nature follow. The work in this chapter is merely formal, no questions of convergency being discussed.

CHAPTER VII: *Complex Theory.* We agree with the authors in regretting that they did not feel able to devote more space to this interesting subject. Bessel's functions are treated in this chapter by means of their definition as definite integrals taken along paths in the complex plane. The functions with pure imaginary argument are here introduced, and the chapter closes with Lipschitz's strict treatment of the semiconvergent expansions already given.

CHAPTER VIII: *Definite Integrals Involving Bessel Functions.* The first half of this chapter is devoted to a clear and well-arranged exposition of the most fundamental parts of this side of the subject which seems to have appealed strongly to the authors. The last half of the chapter deals with the expression of an arbitrary function by means of a remarkable double integral involving Bessel's functions. This integral, which was in substance discovered by C. Neumann, belongs to an extended class of integrals of which Fourier's integral may serve as a type. In fact this integral bears precisely the same relation to the expansion in terms of Bessel's functions considered in Chapter VI. that Fourier's integral bears to Fourier's series.‡ This fact is unfortunately not brought out by the authors although the similarity to Fourier's integral is mentioned.

CHAPTER IX: *The Relation of the Bessel Functions to*

* Liouville's Journal, vol. I.

† See, for instance, Riemann-Hattendorf's *Partielle Differentialgleichungen*, p. 266-269, where, however, only the simplest case is considered and the presentation is somewhat faulty.

‡ This analogy seems the most natural, at least for a treatise on Bessel's functions. It is, however, also true, as has been pointed out by C. Neumann and Mehler, that one may regard this integral as having the same relation to the ordinary development in spherical harmonics, that Fourier's integral bears to Fourier's series.

*Spherical Harmonics.* In view of the physical tendencies of the book it might perhaps have been well, instead of giving a purely analytical treatment of this subject, to develop it in connection with the fact that cylinder coördinates may be regarded as a limiting case of polar coördinates in space. The treatment given, however, brings out clearly the analytic relations.

Before going on to the portion of the book in which the applications of Bessel's functions are taken up let us look back at the theoretical chapters we have just been considering in a somewhat more comprehensive way than we have yet done. While the treatment is too incomplete and even at times inaccurate to satisfy the pure mathematician, it is well adapted to the object the authors had in view of giving to the physicist a working knowledge of the more important properties of Bessel's functions. Even to the student of pure mathematics these chapters give, on account of the numerous sides of the subject on which they touch, an introduction to the theory of Bessel's functions which may fairly be considered satisfactory when supplemented by a certain amount of collateral reading. The presentation is clear and interesting and the references to the original memoirs useful.

The writers have avoided giving undue prominence to one side of the theory at the expense of other sides. There is, however, one matter of no little importance as it seems to us which the authors have hardly touched upon, namely, the differential equation which Bessel's functions satisfy. This equation to be sure is the starting point in Chapter II, but, when once its solutions have been expanded into series, these series, or the definite integrals which are proved equal to them, are made use of whenever any property of the functions is to be established.* Many of the properties of Bessel's functions may, however, be derived with ease and elegance directly from the differential equation. For this purpose one of the binomial forms (line 4, Chapter II, and (81) p. 35) is usually most convenient. The formulæ on pages 16 and 53 might well have been obtained in this way, while perhaps the most important matter which could best be so treated is the subject of the roots of Bessel's functions and allied questions. It should be borne in mind in introducing a student to the study of Bessel's functions that these are only an example of functions which occur frequently in mathematical physics, and which satisfy linear

*Slight exceptions to this statement will be found in Chapters IV, and VI.

differential equations of the second order. With the exception of a few simple cases, which like Bessel's functions can be represented by definite integrals and simple series, no other means is usually available for discussing the properties of these functions, than the differential equation which they satisfy. It seems therefore advisable that in the case of Bessel's functions, although other methods are more or less applicable, the differential equation should not be altogether neglected.

Another matter, a brief treatment of which might well have been included in this treatise, is the expression for the ratio of Bessel's functions of orders $n$ and $n+1$ as a continued fraction.

Passing now to the chapters in which the applications to physical problems are considered, we are struck throughout by the very substantial knowledge of mathematical physics assumed on the part of the reader. This contrasts with the small amount of pure mathematics assumed in the earlier chapters. The simpler problems involving Bessel's functions, such for instance as are taken up in Byerly's text book on *Fourier's Series*, are usually omitted on the ground that the method of treatment is obvious. Here, as in the case of the earlier part of the book, the great diversity of subjects treated, borrowed from German as well as English writers, is an excellent feature.

CHAPTER X: *Vibrations of Membranes.* The free vibrations of circular membranes are here very briefly treated. Many interesting questions connected more or less closely with this subject and involving the use of Bessel's functions are not even mentioned, owing doubtless to the excellent treatment in Lord Rayleigh's *Theory of Sound*. It may be mentioned that formula (21) at the close of the chapter reduces at once by means of the relation

$$J_n(x)\,Y'_n(x) - Y_n(x)\,J'_n(x) = \frac{1}{x},$$

to the form:

$$L = \frac{\pi}{2x^2[J_n(xa)]^2}\left(\frac{1}{[Y_n(xb)]^2} - \frac{1}{[Y_n(xa)]^2}\right).$$

CHAPTER XI: *Hydrodynamics*, opens with a tripos problem concerning vortex motion in which the lines of flow lie in planes passing through an axis. It does not look as though the authors had noticed that the transcendental equation (10) for determining $n$ has no real roots. In order to avoid imaginaries, the functions $J_1$, $Y_1$, and cosh should have been used instead of $I_1$, $K_1$ and cos. There fol-

lows a sketch of an investigation by Lord Kelvin on the small oscillations of a cylindrical vortex about a state of steady motion, and on waves in a cylindrical tank. The chapter closes with a treatment of the two dimensional motion of a viscous liquid which is of peculiar interest as introducing for the first time in the book Bessel functions whose arguments are neither real nor pure imaginary, but of the form $a\sqrt{i}$ where $a$ is real.

CHAPTER XII: *Steady Flow of Electricity or of Heat in Uniform Isotropic Media.* While the problems considered in the preceding chapter led to developments in series of the sort considered in chapter VI, those considered in the present chapter lead to developments not in series, but in definite integrals. The first problem taken up, while simple, is typical of the kind of question treated in this chapter, and it will be worth while for us to consider it in some detail. The problem is, to find the electrostatic potential due to a circular conducting disk charged with electricity. The origin is taken at the centre of the disk and cylinder coördinates $z$, $r$, $\varphi$ are used, the axis of $z$ being perpendicular to the plane of the disk. As the potential is symmetrical with regard to the plane of the disk we need consider only that half of space in which $z$ is positive. The method pursued by the authors is to write down the expression:

$$V=\frac{2c}{\pi}\int_0^\infty e^{-\lambda z}\sin(\lambda r_1)J_0(\lambda r)\frac{d\lambda}{\lambda},$$

$c$ being the potential on the disk and $r_1$ the radius of the disk, and then to verify that this is the desired potential.* This method, while perfectly conclusive, is not satisfactory to the reader as it does not put him in a position to solve a similar problem for himself. We may, however, proceed as follows: From the known values of the surface density at points of the plate we have when $z=+0$ and $r<r_1$:

$$\frac{\partial V}{\partial z}=-\frac{2c}{\pi}\frac{1}{\sqrt{r_1^2-r^2}}.$$

When $z=0$ and $r>r_1$, we have from symmetry: $\frac{\partial V}{\partial z}=0$. We need then merely to find a solution of Laplace's equation which satisfies these conditions and vanishes in the proper way at infinity. Since we have circular symmetry about the axis of $z$ we shall try to build up our potential from so-

* This verification is unnecessarily long; formula (13) *p.* 126 and the corresponding formula when $r>r_1$ being all that is necessary.

lutions of the form: $e^{-\lambda z} J_0(\lambda r)$. Since there is no reason to restrict the value of $\lambda$ here we shall naturally form from these particular solutions not a series but an integral:

$$V = \int_0^\infty a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

From this we get:

$$\frac{\partial V}{\partial z} = -\int_0^\infty \lambda a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

We have then merely to determine the coefficient $a(\lambda)$ in such a way that:

$$\int_0^\infty \lambda a(\lambda)\, J_0(\lambda r)\, d\lambda = \begin{cases} \dfrac{2c}{\pi} \dfrac{1}{\sqrt{r_1^2 - r^2}} & \text{when } r < r_1, \\ 0 & \text{when } r > r_1. \end{cases}$$

This can be done at once by means of formula (166) *p.* 80, which gives:

$$a(\lambda) = \frac{2c}{\pi} \int_0^{r_1} \frac{\rho}{\sqrt{r_1^2 - \rho^2}} J_0(\lambda\rho)\, d\rho.$$

This definite integral can be evaluated (cf. Sonine, Math. Ann. Bd. 16 *p.* 36 bottom) and gives:

$$a(\lambda) = \frac{2c}{\pi} \frac{\sin(\lambda r_1)}{\lambda}.$$

Thus we are led to the value of $V$ given above.

Further problems of a similar, though more complicated nature are treated in the remainder of this chapter which follows closely a paper by H. Weber. Many readers will doubtless wish that the authors had seen fit to devote a little more space to Riemann's treatment of Nobili's rings, as the original is far from easy reading.

CHAPTER XIII: *Propagation of Electromagnetic Waves along Wires.*—As a starting point for this recently developed and already very important subject the equations of the electromagnetic field are first given in general form, and then so specialized as to fit the case in which the waves are guided by a straight wire. $x$ being the distance measured along the wire, $\rho$ the distance from the wire, and $t$ the time, the cases considered are those in which the forces are the real parts of expressions of the form:

$$f(\rho)\, e^{(mz - nt)i}$$

$n$ being real, but $m$, in general, complex. The Bessel's func-

tions which are thus introduced have complex arguments. The methods of work are in the main, those of J. J. Thomson and Hertz. No questions involving developments in series are taken up.

CHAPTER XIV: *Diffraction of Light* contains rather elaborate investigations, chiefly due to Lommel, of the diffraction produced by a small circular opening in a screen, in the course of which interesting numerical results accompanied by diagrams are given. Bessel's functions are here introduced in a way different from anything that has gone before (except the treatment of Kepler's equation), occurring directly as definite integrals and not as elsewhere as the solution of a differential equation. Some extensive analytical investigations find place in this chapter which might better, we think, have been placed in the earlier theoretical part of the book.

CHAPTER XV: *Miscellaneous Applications*, though short, is of decided interest. We choose for special mention from among the half dozen questions here considered the following beautiful problem due to Greenhill: a circular elastic cylinder of small cross section, (*e. g.*, a knitting needle) is held in a vertical position with its lower end clamped and upper end free; to find the greatest length consistent with stability.

There follow two notes, one on Bessel's functions of the second kind, and one on the formula due to J. McMahon for the calculation of the roots of Bessel's functions. The next fifteen pages are devoted to examples to be worked out by the reader. Among these pure mathematics predominates but physical problems also occur in considerable number.

Last but not least come the numerical tables. Of these the first three are due in their present form to Meissel, while the other three are taken from the reports of the British Association. Table I is a twelve place table of the values of $J_0(x)$ and $J_1(x)$ from $x=0$ to $x=15.5$ at intervals of 0.01. This is merely an enlargement of Bessel's original table. Table II which is here published for the first time gives for positive integral values of $n$ and $x$ the values of $J_n(x)$ from $x=1$ to $x=24$ to eighteen decimal places. Table III gives to sixteen decimal places the first fifty roots of $J_1(x)=0$ with the corresponding values of $J_0(x)$. Table IV gives the complex values of $J_0(x\sqrt{i})$ for real values of $x$ from 0 to 6 at intervals of 0.2. The values are given to nine decimal places. Table V gives the values of $I_1(x)=-iJ_1(ix)$ from $x=0$ to $x=5.1$, at intervals of 0.01 to nine decimal

places. Table VI gives, chiefly to twelve significant figures, the values of $I_n(x)$ for integral values of $n$ from 0 to 11 and for values of $x$ at intervals of 0.2 from 0 to 6. This collection of tables will be highly appreciated by all who have to use Bessel's functions in numerical work. A table similar to table III but giving the roots of the equation $J_0(x)=0$ and the corresponding values of $J_1(x)$ would be a welcome addition in a future edition. Such a table would be useful in the numerous problems involving the development of unity in the interval from zero to $a$ in a series of the form $\sum_{s=0}^{s=\infty} A_s J_0(\lambda_s r)$ where $\lambda_s$ is the $s^{\text{th}}$ root of the equation $J_0(\lambda a)=0$.

A short bibliography which though confessedly incomplete will be found useful, and a drawing of the curves $y=J_0(x)$ and $y=J_1(x)$ close the volume.

Serious misprints seem to be rare. On page 14, however, there is one which deserves mention as it occurs in an important formula. The last term in formula (31) reads $-\sum_1^n \frac{1}{2s}$. The minus sign should be changed to plus.

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HARVARD UNIVERSITY,
March, 1896.

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## MODERN METHODS OF ANALYTICAL GEOMETRY.

*An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry.* By CHARLOTTE ANGAS SCOTT, D. Sc., Girton College, Cambridge; Professor of Mathematics in Bryn Mawr College, Pennsylvania. XII+288 pp., octavo. London and New York, Macmillan and Co. $2.50.

A minor excellence of this book, for which many readers will feel truly grateful, is the fact that it is written in the English of English speaking and writing people. Private abbreviations, cabalistic marks necessitating constant reference to an elusive "list of signs," Teutonisms, and Greek logomachy in the way of "tetrastigms," etc., are agreeably absent. The parvenu "join" is flattered with recognition, but this term is now in such general use that to protest further against it will be of little avail. It is in a measure a consolation that no one is as yet permitted to "enthuse" over this acquisition to the language. "Joining line" is a

letter and a syllable shorter than "intersection," the correlative term. If we say "join" shall we not say "meet" or "cross"? "Double point" also might better be written in full. But these are trivial affairs.

The book modestly designated by the author as an "introduction," and certainly well adapted to the needs of the learner, is in fact a compact scholarly work on the more accessible principles and methods of modern analytical geometry. It exhibits to a marked degree that genial breadth of treatment and conciseness which are associated only with mature scholarship and extensive and accurate information. Everything fits naturally in place, nothing is cramped or forced out of its natural relations. In the limited space of less than three hundred pages the great ideas of geometry, including the doctrine of coördinates, projection and dualism, correspondence, and the absolute, are clearly and skillfully developed, and illustrated by application to an ample list of well-chosen examples, a sufficient number of the latter being reserved as an exercise for the reader. We notice with pleasure that the book is entirely free from the criticism, so often and so properly urged against many English mathematical works, that their authors exhibit no adequate conception of the distinction between a general principle and a particular example. Miss Scott is not only keenly alive to this distinction; she is also perfectly aware of the importance of choosing methods adapted to the subject, which is here primarily geometry, not algebra.

Historical and other references are only rarely given, on the assigned ground of the introductory character of the book. Some years ago a prominent French mathematician offered a different excuse for the same omission, viz., that he could not by giving references enhance the fame of the illustrious geometers who had preceded him. We apprehend that neither plea will be good in court. The beginner as well as the scholar should have every facility for acquainting himself with the history and classical works of his subject. And, as we have pointed out, in the present case "introductory" is not by any means synonymous with "elementary."

The author's mode of presentation, usually good, becomes occasionally very felicitous. At the beginning the justification of the line as an element, co-important with the point, of the plane is well stated. The infinite and imaginary elements—line at infinity, circular points, isotropic lines, etc., are skillfully managed. The chapter on the Ab-

solute is an admirably lucid and elementary exposition. The author aims throughout at generality and elasticity of treatment. Her success in this respect is very pronounced, and is in fact one of the most acceptable features of the book.

The book covers of course to a large extent the same field as the treatises of Salmon to which the author very appreciatively refers, and to which the reader is directed for the study of many aspects of the subject. No mention is however made of the German editions of Salmon's works, which are in some cases greatly superior to the originals. There is ample room for new and modern books in various departments of geometry, and we will trust that Miss Scott's very successful present venture into this field will not be her last.

Pure geometry travels flying light, without baggage, and lives on the country. The analytical arm of the service carries a heavy, but nowadays perfectly mobile, siege train with many supply depots. And first of all the accoutrements in the way of coördinates must be well appointed. A coördinate is any quantity that helps to determine an element, and an element is any geometric form. The simplest elements of plane geometry are the point and the line. On each of these a geometry can be constructed; and these two geometries are identical in substance and differ only in guise. How many coördinates a given element may have is a matter of taste. Cartesian geometry, content with bare necessaries, considers only the point as element and assigns to this two coördinates. With three coördinates for the point, (or line) however, the algebraic evolutions become much more symmetric, inasmuch as all equations are then homogeneous. At the the same time, and this is the real advantage, the whole coördinate system becomes projective, and accompanies the corps of figures to which it is attached in their passage from one plane to another. The three coördinates of point or line are referred to the sides and vertices of a triangle of reference, most conveniently the same triangle in both cases. In the Cartesian geometry, as viewed in this aspect, the coördinate axes are two sides of this triangle of reference, while the third side is the conventional "line at infinity." The latter is a line, geometrically because its projection is a line, and analytically because the equations of all lines infinitely removed tend to the same form. Since it is true that every quadrangle can be projected into every other, it might seem desirable to employ four coördinates instead of three. In fact, however, one vertex of the quadrangle is already in demand as an adjust-

ment—unit point, *vide infra*—for the coördinate system attached to the other three.

If a plane figure is projected into a new plane, the triangle of reference of the original plane becomes a triangle which may be used as a triangle of reference in the new plane. This being the case the equations of the two figures are identical. If, however, it is desired to change the triangle of reference in a plane, this is effected by an integral homogeneous linear transformation of coördinates with non-vanishing discriminant. The equations of transformation admit of a second interpretation, viz.: the triangle of reference may be regarded as fixed while the plane undergoes distortion. In the latter case any figure is distorted into a projective figure. The discussion of the relation between projection and linear transformation is elaborated in a special chapter.

Since two coördinates are sufficient to determine a point or line, the three coördinates actually employed in each case are connected by an identity. The latter is linear for point coördinates, but, curiously enough, for line coördinates it is quadratic. In this violation of the principle of dualism we have an epitome of the one-sided character of our system of metric, leading to no little inconvenience until it finally comes to be recognized as a one-sided limiting form of a more general conception in which the principle of dualism retains its validity in all cases, and at the same time all properties are projective. Properties are then still divided into two classes according as they belong to a figure in itself or express its relations to the absolute conic. Until this step is taken, however, it follows from the identities above mentioned that we have only one singular (infinitely distant) line, but two infinitely distant points. Closely connected with this is the fact that there is in our ordinary geometry no natural unit of length, while there is a natural unit angle, viz., the complete circumference.

A coördinate, as an algebraic quantity, may, and not only may but inevitably will, assume imaginary as well as real values, and its imaginary values are infinitely more numerous than its real ones. The common conception of geometry, therefore, requires to be idealized by the introduction of imaginary *geometric* elements. Writers on analytical geometry do not ordinarily concern themselves with the development of this idea and the author follows the usual practice. Any extensive treatment of the imaginary may well be left to special treatises, but some of the more elementary notions might properly have been mentioned

here. For example, what is called the equation of a plane curve really represents a surface of which the curve is only a section; two such surfaces intersect in a finite number of distinct points; isolated points of a curve are happily explained in this connection. The nature of the continuity of the surface is readily ascertained, and the theories of deficiency and of birational transformation are intimately related to this subject. But this is already going rather far,—beyond the limit that the author has set for the work.

The great unifying and fertilizing principle of geometry is the doctrine of projection, or more generally of correspondence. The great advances which analytic geometry has made during the present century are due entirely to the adoption of projective ideas and methods. With this has gone a recasting of the entire notation of the science into projective form by the aid of anharmonic ratio. And thus finally all essential distinction between synthetic and analytic geometry has long since disappeared. Still our modern works on analytic geometry are very distinctively analytic, and writers on pure geometry, almost without exception, avoid carefully any explicit mention of coördinates. But it is an easy prediction that this fictitious boundary line will not long be observed. The author of the *Introduction* gives liberal recognition to the methods of pure geometry; but we could wish that she had gone still further in this direction. For example an alternative, purely geometrical, proof of the characteristic property of the complete quadrangle and quadrilateral (p. 43) might well have been given. We miss, too, the many easy geometric deductions from the notion of projective ranges and pencils, including ranges on a conic and tangents of a conic, especially in the latter case the geometric proof of Pascal's and Brianchon's theorems, which in point of conciseness and exact adaptation to the subject are simply perfection. One matter in particular is so important and so pertinent to *analytic* geometry, that we cannot but regard its omission as unfortunate. It is well known that the anharmonic ratio is the simplest *invariant*. It is also true that the anharmonic ratio is a *universal coördinate*, all other coördinates being merely disguised forms of anharmonic ratios. In fact, the anharmonic ratio is referred to three elements at which it takes the values $\infty$, 1, 0, respectively, and if any coördinate $x$ of an element $P$ takes these three values at the elements, $A$, $B$, $C$, then $x=(ABCP)$. For example, the Cartesian coördinate $x$ of a point on a line is an anharmonic ratio referred to the point at infinity $A$, the origin $C$ and the unit point $B$, $CB$

being the arbitrary unit of length. Again the ratios of the trimetric coördinates of a point, $P$, referred to a triangle, $ABC$ are

$$\frac{x_1}{x_2}=(C-ADBP),\ \frac{x_2}{x_3}=(A-BDCP),\ \frac{x_3}{x_1}=(B-CDAP),$$

when $D$ is the unit point, *i. e.*, the point for which $x_1 : x_2 : x_3 = 1$. On this principle depends the possibility of making analytic geometry thoroughly projective, the coördinates themselves admitting of projective definition.

Geometric properties are primarily classified as projective and non-projective. Beside these the terms "descriptive" and "metrical" are in common use and are given prominence by the author. According to Cayley (Salmon's Higher Plane Curves, Chapter I) properties are descriptive if they have no reference to the line at infinity or the circular points; otherwise they are metrical. In this use "descriptive" is identical with "projective" and "metrical" with "non-projective." The author's definition of "descriptive" is the analytical equivalent of Cayley's, but this definition is not strictly adhered to inasmuch as properties expressed in terms of anharmonic ratios are afterward described as metrical (p. 147). It seems to us that the term "descriptive" which is commonly employed in conjunction with "projective" and non-projective, as a species of *tertium quid*, might well be reserved to designate the technical art to which it was originally applied. "Metrical" has an intrinsic value, and in the sense of "involving measurement" could hardly be dispensed with. That its use is however liable to be attended with some ambiguity is evidenced by the statement (p. 147). "Theorems stated with reference to cross-ratio are metric theorems in projective form," which to our mind rather gains in force if the words "metric" and "projective" are interchanged.

In pure geometry projection is a matter complete in itself, but in analytic geometry it has its counterpart in the theory of linear transformation. The latter subject may be fairly said to lie at the very center of modern mathematics. From the formal standpoint projective geometry is merely an interpretation in space of linear transformation, an interpretation so interesting and valuable that when, with an increased number of variables in the equations, ordinary space no longer suffices for the purpose, no hesitation is felt in positing a hyper-space with any number of dimensions. In fact the study of geometry has, for many, its chief value in the fact that it furnishes an abundance of simple and current il-

lustrations of principles, whose abstract foundation belongs to other more difficult mathematical sciences. For example, as the author very appropriately points out, all the collineations of a given space form a *group*. This group has a variety of *subgroups*, and it has also its peculiar *invariants* and *covariants*.

The chapter on Projection and Linear Transformation is followed by one on Correspondence, in which the general notion of correspondence is explained and applied to the general (1, 1) quadric correspondence, to reciprocation, and briefly to the birational transformation of a curve into itself. Following this is the chapter on the Absolute, and finally a very short but well-written introduction to Invariants and Covariants. Descriptive and metrical properties, and the theory of anharmonic ratio and involutions are treated at length, but compactly in the earlier chapters. There are also separate chapters on Dualism and on Unicursal Curves.

The book will be found to gain continually in interest on repeated reading, and this is due especially to its eminently suggestive character. The author succeeds admirably in conveying to the reader, if he be fit, an accurate knowledge and command of general principles. We know of no introductory work which is better adapted in this particular for the use of those who desire not merely to learn but also to master geometry.

F. N. COLE.

COLUMBIA UNIVERSITY,
*March*, 1896.

# ON A REMARKABLE COVARIANT OF A SYSTEM OF QUANTICS.

BY PROFESSOR H. B. NEWSON.

I wish to call attention in this paper to a remarkable covariant of a system of $n$ quantics in $n$ homogeneous variables. This covariant although not entirely new has not received the attention which it deserves. While the exposition here given is limited to the case of three quantics in three homogeneous variables, it will be seen that the methods are applicable to a system of $n$ quantics in $n$ variables. This covariant is here approached from the geometric point of view and the language used is that of the theory of higher plane curves. I shall attempt only to sketch the work in outline and shall omit all details as unsuitable to the place and object of this paper.

The covariant in question bears much the same relation to the Jacobian of the system of quantics as the Steinerian of a single quantic bears to the Hessian. In order therefore to get the most advantageous starting point for the discussion, we shall begin by restating two fundamental properties of the Jacobian.

Let there be given three curves, $U$, $V$ and $W$, whose degrees are respectively $m$, $n$, and $p$. The Jacobian of these three curves, whose equation is obtained by equating to zero the functional determinant of $U$, $V$ and $W$, has two well known polar properties as follows:

(1). *The Jacobian of $U$, $V$ and $W$ is the locus of all points common to the first polars of a point $(x', y,' z')$ with respect to $U$, $V$ and $W$.*

(2). *The Jacobian of $U$, $V$ and $W$ is also the locus of the point $(x', y', z')$, whose polar lines with respect to $U$, $V$ and $W$, meet in a point.*

The equation of the Jacobian is obtained in two different ways, each of which leads to one of the properties in question. The first polars of $(x', y', z')$ with respect to $U$, $V$ and $W$, are given by the following equations, using Salmon's notation (see Higher Plane Curves, Art. 61).

$$(1) \qquad \begin{aligned} x'U_1+y'U_2+z'U_3&=0,\\ x'V_1+y'V_2+z'V_3&=0,\\ x'W_1+y'W_2+z'W_3&=0. \end{aligned}$$

When these three equations are simultaneous their resultant vanishes, and we have the equation of the Jacobian and the proof of property (1).

The polar lines of $(x', y', z')$ with respect to $U$, $V$ and $W$ are given by the equations

$$(2) \qquad \begin{aligned} xU_1'+yU_2'+zU_3'&=0,\\ xV_1'+yV_2'+zV_3'&=0,\\ xW_1'+yW_2'+zW_3'&=0. \end{aligned}$$

When these three lines meet in a point the resultant of their equations vanishes, and we have again the equation of the Jacobian and the proof of the second property.

The Jacobian is a curve of order $(m+n+p-3)$.

But instead of confining our attention to the Jacobian we might also consider the locus of the point $(x', y', z')$ whose first polars meet in a point. The equation of this locus is obtained by eliminating $x$, $y$ and $z$ from equations (1). It is easy to see that we obtain the same equation by eliminating $x'$, $y'$ and $z'$ from equations (2), except that $x$, $y$, $z$ are interchanged with $x'$, $y'$ and $z'$.

I propose to name this locus provisionally the Cremonian of $U$, $V$ and $W$; I say provisionally because I do not know by whom the curve was first studied. It is specifically mentioned in Cremona's *Introduzione ad una teoria geometrica delle curve piane*, Art. 98. See also Hagen's *Synopsis der Hoeheren Mathematik*, vol. 2, page 201. In Higher Plane Curves, Art. 401, Salmon mentions this locus which is here called the Cremonian, but says: "We shall confine ourselves to the consideration of the case when the three curves are first polars of a given curve, in which case the Jacobian in the Hessian of that curve, and the other locus now mentioned is its Steinerian." It seems to me that the general case is too important to be dismissed in this way; in fact more important than the special case which is frequently mentioned.

The Cremonian has two important geometric properties corresponding to the two methods by which its equation may be obtained.

(1). *The Cremonian of $U$, $V$ and $W$ is the locus of the point $(x', y', z')$ whose first polars with respect to $U$, $V$, and $W$ have a common point; the locus of these common points is of course the Jacobian.*

(2). *The Cremonian of $U$, $V$, and $W$ is also the locus of $(x, y, z)$ the point of intersection of the polar lines of $(x', y', z')$, with respect to $U$, $V$ and $W$; i. e., it is the locus of the point of intersection of the polar lines of the points on the Jacobian.*

If we select some point as $C$ on the Cremonian, its first polars intersect in a point $J$ on the Jacobian; the polar lines of the point $J$ meet in a point on the Cremonian. It

is not difficult to see that this last point must be the point $C$. Thus there are two relations connecting the points $C$ and $J$, which may well be called corresponding points on the two curves. All the properties of the Jacobian and the Cremonian so far mentioned may be summed up in the following comprehensive theorem :

*If the first polars of a point C with respect to three curves U, V and W meet in a point J; then the polar lines of the point J with respect to the same three curves meet in the point C; and vice versa. The locus of all points C is the Cremonian of the three curves, and the locus of all points J is the Jacobian of the three curves.*

When we consider the process of eliminating $x$, $y$ and $z$ from the three equations (1), it is easy to see that the Cremonian is of order $(mn+mp+np-2m-2n-2p+3)$. And since the two curves have a one to one correspondence, we infer that the Cremonian and the Jacobian have the same deficiency.

The lines joining corresponding points on the Jacobian and Cremonian, envelope a curve for which I have no better name than the Hyper-Cayleyan; because it is generated in a manner similar to the method of generating the Cayleyan. It is evident that the Hyper-Cayleyan has a one to one correspondence both with the Cremonian and the Jacobian, and has therefore the same deficiency. The class of this Hyper-Cayleyan is, we know, equal to the sum of the orders of the Jacobian and the Cremonian, *i. e.*, it is given by the number $(mn+mp+np-m-n-p)$.

It is not a difficult task to prepare a table of the Plueckerian characteristics of the three curves, the Jacobian, the Cremonian and the Hyper-Cayleyan, similar to that for the Hessian, Steinerian and the Cayleyan given in Clebsch-Lindemann's *Vorlesungen ueber Geometrie*, vol. I., page 368. This table is also given in Hagen's *Synopsis der Hoeheren Mathematik*, vol. II., page 203.

In the most general case we suppose that the three curves, $U$, $V$, $W$, have no double points, are of different degrees, $m$, $n$, $p$, and have no points or points common to all three. A study of the special cases brings to light a large number of interesting theories. Thus we may consider the cases where one or more of the curves have double points, cusps or multiple points: also the cases where two and then three of the curves have points in common, which may be ordinary points on the curve or multiple points of various orders.

When $U$, $V$, $W$ are all of the same degree, we know that the Jacobian of the three curves is the locus of the double

points on all the curves of the system $U+kV+lW=0$. When $U$, $V$, $W$ are all of the same degree and at the same time are first polars of three points with respect to a curve $C$ of degree $(m+1)$ or $n$; then the Jacobian of $U$, $V$, $W$ becomes the Hessian of $C$, the Cremonian of $U$, $V$, $W$ becomes the Steinerian of $C$, and the Hyper-Cayleyan of $U$, $V$, $W$ becomes the Cayleyan of $C$.

It is not difficult to extend the conception of the Cremonian here developed to spaces of higher dimensions. In three dimensions we must have given four surfaces, $U$, $V$, $W$, $T$; the Jacobian and Cremonian are then defined and their equations obtained in a manner analogous to that employed in two dimensions. The lines joining the corresponding points on the Jacobian and Cremonian surfaces form a system of $\infty^2$ lines in space. I have not examined the properties of this system of lines.

In a paper published in the *Kansas University Quarterly* vol. 3., No. 2, entitled *On the Hessian, Jacobian, Steinerian, etc., in Geometry of One Dimension*, the writer has developed some of the properties of the Cremonian of two binary quantics. In that paper the name Cremonian was not used; but the covariant in question was designated by the symbol $M(VW)$.

So far as my knowledge goes there has been no systematic and detailed study of the properties of the covariants, which are here called the Cremonian and Hyper-Cayleyan. It seems to me that they are worthy of serious investigation, and it is to be hoped that some one will undertake the task.

KANSAS STATE UNIVERSITY,
*February* 17, 1896.

## NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, March 28th, at three o'clock, the President, Dr. HILL, in the chair. There were fifteen members present. Two nominations for membership were received. The following papers were read:

(1) Professor H. B. NEWSON: "On a remarkable covariant of a system of $n$ quantics."

(2) Professor E. W. BROWN: "On the particular integrals of linear differential equations, and their application to the lunar theory."

In the absence of Professor NEWSON, his paper was read by the Secretary.

THE Secretary of the Society has been directed by the Council to prepare a new List of Members for early publication, and has accordingly issued a circular asking for necessary information from the members of the Society. Those members who have not already responded to this circular are requested to do so at once.

A SUMMER meeting of the Society will be held at Buffalo on Monday and Tuesday, August 31st and September 1st, in affiliation with the American Association for the Advancement of Science. Members intending to contribute papers or to be present are requested to notify the Secretary.

FACULTÉ DES SCIENCES. During the second semester of the current academic year, the following mathematical courses are offered at the Sorbonne, in Paris:—Professor Hermite: Theory of Eulerian integrals and the theory of elliptic functions;—Professor Picard: Theory of twisted curves and of algebraic surfaces;—Professor Appell: General laws of the motions of systems, analytical mechanics, hydrodynamics;—Professor Wolf: Astronomical subjects included in the programme of the *Licence*;—Professor Poincaré: Optics;—Professor Boussinesq: Theory of oscillatory waves (rise and fall of the sea; waves produced on the surface of still water by the immersion of a solid, or by superficial impulsion like that of an air current, etc.). The following supplementary course is also given: M. Painlevé: Differential and integral calculus. Mathematical conferences are conducted by Messrs. Painlevé, Puiseux, Raffy, Andoyer and Blutel.

HARVARD UNIVERSITY. During the year 1896–97, Professor Asaph Hall (U. S. Navy) will give a course in Celestial Mechanics. The following advanced mathematical courses are also offered:— By Professor J. M. Peirce: Algebraic Plane Curves; Quaternions (second course); Special Developments of the Calculus of Quaternions †.— By Professor Byerly; Differential and Integral Calculus (second course); Analytic Mechanics (with Mr. Whittemore); Rigid Dynamics †.—By Professors Byerly and B. O. Pierce: Fourier's Series, Spherical Harmonics and Potential Function.—By Professor B. O. Pierce: Theory of Surfaces †.—By Professor Osgood: Infinite Series and Products †; Elliptic Functions.—By Professor Bôcher: Theory of Equations and Invariants †; Modern Geometry; Theory of Numbers †; Theory of Functions (first course); Bessel's Functions †.

The above courses will each consist of three lectures a week throughout the entire academic year except those marked † which consist of at least one-half that number of lectures. The following reading courses are also offered:— By Professor J. M. Peirce: Studies in Mathematical Literature.—By Professor B. O. Peirce: Elasticity.—By Professor Osgood: Klein's Ikosaeder.

In addition to the above the mathematical department also offers instruction in the following elementary subjects: Solid Geometry, Algebra, Logarithms, Plane and Spherical Trigonometry, Analytic Geometry of two and three dimensions, Differential and Integral Calculus, Mechanics.

THE UNIVERSITY OF CHICAGO. The mathematical courses to be offered during the summer quarter of 1896 have already been announced in the BULLETIN (p. 183).

During the three quarters (*a*, *w*, *s*) of the academic year 1896–97 the following advanced mathematical courses (four or five hours weekly) will be offered: By Professor Moore, Seminar devoted to research work and Kronecker's arithmetical theory of algebraical quantities (*a*, *w*), Theory of groups (*w*), Projective geometry (*a*); by Professor Bolza, Galois theory of algebraic equations (*s*), Solid analytics (*s*); by Associate Professor Maschke, Modern analytic geometry (*a*), Higher plane curves (*w*), Algebraic surfaces (*s*), Theory of functions of a complex variable (*s*), Advanced integral calculus (*a*, *w*); by Dr. Young, Theory of numbers (*w*); by Dr. Boyd, Differential-geometry of surfaces (*a*); by Dr. Hancock, Calculus of variations (*s*), Theory of equations (*a*, *w*); by Dr. Laves, Analytic me-

chanics (*a*, *w*); by Mr. Slaught, Differential equations (*s*). The Mathematical Club meets fortnightly.

THE Council of The AMERICAN MATHEMATICAL SOCIETY has given its approval to a proposition for holding a colloquium in connection with the summer meeting at Buffalo next August. It is thought by many that the benefits arising from the summer meeting will be greatly increased, if those attending have an opportunity of hearing brief accounts of the progress made in different branches of the higher mathematics. It is intended to have two or three short courses of lectures by eminent mathematicians, every lecture being followed by an informal discussion. Any arrangements that may be made will be announced in a later number of the BULLETIN.

A NEW American mathematical journal is soon to appear. It will be edited by Professor W. E. Story of Clark University, Worcester, Mass., and will be issued bimonthly. Each number, it is said, will contain, in addition to papers devoted to mathematical research, notices of the transactions of learned societies and the titles of recent mathematical books and articles. Each number will consist of about ninety-six large octavo pages. The subscription price will be $5.00 per year.

WE take pleasure in announcing the appearance of the volume of *Mathematical Papers read at the International Mathematical Congress held in connection with the Columbian Exposition, Chicago*, 1893. The publishers are Macmillan and Co. The price of the volume is $4.00.

A Mathematical work of unusual character is announced for July next, by John Wiley and Sons. It is entitled *Higher Mathematics for Engineering Colleges*, and is edited by Professor Mansfield Merriman and Professor R. S. Woodward. The volume is intended for advanced students in engineering colleges and other institutions, and is designed to supplement their elementary mathematics by a knowledge of the elements of such branches of modern higher mathematics as admit of practical application. In addition to chapters by the editors on the *Solution of Equations* and *Probabilities and Theory of Errors*, the work will contain the following chapters: Professor W. E. Byerly, Harvard University, *Harmonic Functions*; Professor Thomas S. Fiske, Columbia University, *Functions of a Complex Variable*; Professor G. B. Halsted, University of Texas, *Projective Geometry*; Professor E. W. Hyde, University of Cincinnati, *Point Analysis and Aus-*

*dehnungslehre*; Professor W. W. Johnson, U. S. Naval Academy, *Differential Equations*; Professor A. Macfarlane, Lehigh University, *Vector Analysis and Quaternions*; Professor James McMahon, Cornell University, *Hyperbolic Functions*; Professor F. Morley, Haverford College, *Elliptic Integrals and Functions*; Professor D. E. Smith, Michigan Normal School, *History of Modern Mathematics*; Professor L. G. Weld, University of Iowa, *Determinants*.

ACCORDING to *Nature* the French Association for the Advancement of Science held a very successful meeting at Tunis beginning April 1st. More than four hundred members were present. The principal streets of Tunis were elaborately decorated in honor of the occasion, and, as hotel accommodation was limited, the Lyceé Carnot was placed at the disposal of visitors. The association met in Algiers fifteen years ago, so that this was the second meeting on African soil. Next year's meeting will take place at St. Etienne.

WE learn from the *Revue générale des Sciences pures et appliquées* that Professor Mittag-Leffler, founder of the *Acta Mathematica*, completed last March his fiftieth year. On that occasion a number of his admirers presented him with a congratulatory address written in four languages, German, English, French and Italian, and his own portrait done in oil by the Finnish painter Edelfelt.

AT the April meeting of the National Academy of Sciencs Professor R. S. Woodward, of the department of mechanics in Columbia University, was elected to membership.

## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

ALBERT (G.). Die Platonische Zahl und einige Conjecturen zu Platon sowie zu Lukrez. Wien, 1896. 8vo. 1 plate. Mk. 1.00.

BOREL (E.). See NIEWENGLOWSKI (B.).

CARONNET (T.). Problèmes de licence ès sciences mathématiques donnés à la Sorbonne. Avec les solutions développées. (Année 1895, sessions de juillet et de novembre, élèves libres et élèves de l'École normale.) Paris, Nony, 1895. 8vo. Fr. 1.00

CARVALLO (E.). Méthode pratique pour la résolution numérique complète des équations algébriques ou transcendantes. Paris, Nony, 1896. 4to. Fr. 1.50

EDWARDS (J.). An elementary treatise on the differential calculus, with applications and numerous examples. 3d edition, revised and enlarged. London, Macmillan, 1896. 8vo. 546 pp. 14s.

GALOPIN-SCHAUB (C.). Premières notions du calcul des quaternions. Bâle, 1896. 12mo. 64 pp. Mk. 1.50

GOURSAT (E.). Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes. Vol. I: Problème de Cauchy ; caractéristiques ; intégrales intermédiaires. Paris, Hermann, 1896. 8vo. 8 and 226 pp. Fr. 7.50

GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Band I, Theil 2: Die Ausdehnungslehre von 1862; in Gemeinschaft mit H. Grassmann dem Jüngeren herausgegeben von F. Engel. Leipzig, Teubner, 1896. 8vo. 8 and 512 pp. Mk. 16.00

HAAS (A.). Beiträge zur graphischen Darstellung der höheren Plankurven. [Progr.] Stuttgart, 1895. 4to. 2 and 52 pp.

KOENIGSBERGER (L.). Hermann von Helmholtz's Untersuchungen über die Grundlagen der Mathematik und Mechanik. Mit einem Bildniss Hermann von Helmholtz's. Leipzig, Teubner, 1896. 8vo. 6 and 58 pp. Mk. 2.40

MARTIN (ARTEMAS.). About cube numbers whose sum is a cube number. [*Mathematical Magazine*, Vol. 2, pp. 153–160 and 185–190.] 4to.

——— About biquadrate numbers whose sum is a biquadrate. [*Mathematical Magazine*, Vol. 2, pp. 173–184.] 4to.

MASCHKE (H.). Ueber die Darstellung endlicher Gruppen durch Cayley'sche Farbendiagramme. [*Nachrichten der K. Gesellschaft der Wissenschaften zur Göttingen*, math.-phys. Klasse, 1896, Heft 1.] 8vo. 6 pp.

MATHEMATICAL QUESTIONS AND SOLUTIONS from the *Educational Times*. Vol. 64. Edited by W. J. C. Miller. London, Hodgson, 1896. 8vo. 128 pp. Boards. 6s. 6d.

MILINOWSKI (A.). Elementar-synthetische Geometrie der Kegelschnitte. Mit einem Anhange über die gleichseitige Hyperbel. 2te Ausgabe (reduced in price). Leipzig, Teubner, 1896. 8vo. 12 and 412 pp., 10 and 135 pp. Mk. 4.00

MILLER (G. A.). Sur les groupes de substitutions. [*Comptes rendus*, Vol. 122, pp. 370–372]. 4to.

MILLER (W. J. C.). See MATHEMATICAL QUESTIONS AND SOLUTIONS.

NIEWENGLOWSKI (B.). Cours de géométrie analytique. (En 3 volumes.) Vol. III.: Géométrie dans l'espace, avec une note sur les transformations en géométrie, par E. Borel. Paris, Gauthier-Villars, 1896. 8vo. 1er fascicule: pp. 1–336. Complete in two fascicules. Fr. 13.00

PORCHON (P.). Compléments d'algèbre et de géométrie: applications de l'algèbre à la géométrie, sections coniques; rédigés conformément aux derniers programmes officiels pour la classe de première (sciences). Paris, Alcan, 1894. 16mo. 204 pp.

SCHLEGEL (V.) Die Grassmann'sche Ausdehnungslehre. Ein Beitrag zur Geschichte der Mathematik in den letzten fünfzig Jahren. Schluss. (*Zeitschrift für Mathematik und Physik*, Vol. 41 (1896), historisch-literarische Abtheilung, pp. 41–59.) 8vo.

SEGRE (C.). Intorno ad un carattere delle superficie e delle varietà superiori algebriche. [*Accademia reale delle scienze di Torino*, Anno 1895–96.] Torino, Clausen, 1896. 8vo. 20 pp.

## II. ELEMENTARY MATHEMATICS.

ADSUAR Y MORENO (J.). Ejercicios prácticos de geometria. Parte I: Cuestionario. Madrid, 1895. 8vo. 161 pp.

BUNKOFER (W.) Die arithmetischen Funktionen der drei ersten Ordnungen. Methodische Studie zum Ausbau der Gymnasialmathematik. [Progr.] Wertheim, 1895. 4to. 18 pp.

CAPELO (J.). Tratado de algebra elemental, conforme al programa oficial. 2a edicion. Paris, Galland, 1896. 8vo. 192 pp.

CHAMBERS (W.). Mathematical tables, edited by J. Pryde. New edition. New York, Van Nostrand, 1896. 12mo. Cloth. $1.75

COMBEROUSSE (C. DE). See ROUCHÉ (E.) et COMBEROUSSE (C. DE).

DARIÈS (G.). Mathématiques. (Bibliothèque du conducteur de travaux publiques.) Paris, Dunod, 1896. 16mo. 8 and 350 pp.

FISCHER (I. G.). Leitfaden zum Unterricht in der Elementar-Geometrie. 2ter Kursus. 12te Auflage, neu durchgesehen von C. Vogel. Halle, Gesenius, 1896. 8vo. 44 pp. Boards. Mk. 0.60

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 47ste Auflage. Halle, 1896. 8vo. 166 and 35 pp. Half-morocco. Mk. 2.50

——— Fünfstellige logarithmische und trigonometrische Tafeln. Kleine Ausgabe. 6te Auflage. Halle, 1896. 8vo. 4 and 96 pp. Cloth. Mk. 1.60

JUEL (C.). Elementaer Stereometri. Kjöbenhavn, 1896. 8vo. 148 pp. Mk. 3.00

LENGAUER (J.). Die Grundlehren der Stereometrie. Ein Leitfaden für den Unterricht, mit Uebungsaufgaben. Kempten, Kösel, 1896. 8vo. 3 and 111 pp. Mk. 1.50

MARTIN (P.) und SCHMIDT (O.). Raumlehre für Mittelschulen, Bürgerschulen und verwandte Anstalten. Nach Formengemeinschaften bearbeitet. Heft 1: Der Wohnort. Dessau, Kahle, 1896. 8vo. 8 and 80 pp. Boards. Mk. 0.60

PRYDE (J.). See CHAMBERS (W.).

RAJOLA-PESCARINI (L.). Lezioni di geometria piana. Parte II. Napoli, 1896. 12mo. 112 pp. Fr. 2.00

RICHARD (L.). Sténarithmie Richard, ou l'art de calculer aussi vite que la pensée. 18e édition. Paris, librairie de l'École pratique de commerce, 1896. 8vo. 64 pp. Fr. 2.00

ROUCHÉ (E.) et COMBEROUSSE (C. DE). Leçons de géométrie, rédigées suivant les derniers programmes officiels et accompagnées pour chaque leçon d'exercices et de problèmes gradués. (En 4 volumes.) Vol. I: La ligne droite et la circonférence de cercle. Paris, Gauthier-Villars, 1896. 8vo. 10 and 168 pp. Fr. 2.75

———. Solutions détaillées des exercices et problèmes énoncés dans les Leçons de géométrie. (En 4 parties.) Partie I. Paris, Gauthier-Villars, 1896. 8vo. Fr. 3.00

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